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A CRITERION FOR THE EXISTENCE OF INVARIANT PROBABILITY MEASURES IN MARKOV PROCESSES

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A criterion for the existence of invariant probability measures in Markov processes *)

by

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Summary

In this paper we investigate the existence of invariant probability measures for Markov processes on noncompact state spaces. The introduced criterion is a generalization of a Foster criterion [Foster (1953), theorem 2] and of a Liapunov function criterion [Kushner (1971), section 8.6.5]. As an illustration of the applicability of our criterion, we show that it is satisfied for the Lindley model in queueing problems.

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**) Universitaire Instelling Antwerpen. Part of this work was done while the second author was an "aspirant navorser N.F.W.O."

1. PRELIMINARIES

Let (X_n), $n=0,1,2,\ldots$ be a Markov process defined on the probability space $(\Omega,\mathcal{A},P)$. We assume that the state space E is a separable metric space with Borel σ-algebra $\mathcal{F}$.

For f a measurable function on E we denote

$$Pf(x) = \int_E P(x,dy)f(y),$$

where $P(x,B)$, $x \in E$ and $B \in \mathcal{F}$, is the stochastic kernel corresponding to the Markov process ($P(x,E)=1$ for all $x \in E$). Further, given any set B and function f we write $I_B f$ for the function which equals f on B and is zero on B^c (the complement of set B). We define

$$P_B^n = (PI_{B^c})^{n-1}P \ , \ n \geq 1,$$

where the $(n-1)^{th}$ power means that the operator PI_{B^c} is applied $(n-1)$ times ($(PI_{B^c})^0 = I$ the identical operator).

Finally we introduce

$$G_B = \sum_{n=1}^{\infty} P_B^n.$$

When applying one of the above introduced operators on some function f it is tacitly assumed that the operator acted on the function $|f|$ gives a function which is everywhere finite (for example when we write Pf then it is assumed that

$$\int_E P(x,dy)\ |f(y)| < \infty \text{ for all } x \in E).$$

For μ a measure and g a measurable function we denote

$$\mu g = \int g(x)\mu(dx).$$

2. INVARIANT PROBABILITY MEASURES

A measure μ is called invariant for the Markov process (X_n), $n=0,1,...$ with kernel $P(x,B)$, $x \in E$ and $B \in \mathcal{F}$, if

$$\mu = \mu P = \int_E \mu(dx)\ P(x,.)$$

The following conditions are sufficient for the existence of an invariant probability measure

S) Stability condition

There exists a compact set A *and a finite nonnegative and measurable function* $\phi(x)$ *such that*

$$1 + PI_{A^c}\ \phi(x) \le \phi(x) \qquad \textit{for all } x \in A^c$$

and

$$PI_{A^c}\ \phi(x) \qquad \textit{is bounded on } A.$$

C) *If* $f \in C(E)$ *(the class of real valued, bounded and continuous func-*

tions on E), *then* $Pf \in C(E)$ *and* $PI_A f \in C(E)$.

We note that the first part of condition C is equivalent to the assumption that the Markov process is stable [Loève (1960), p. 623]. A sufficient condition for the second part of assumption C is $P(x,A\backslash A^i) = 0$ for all $x \in E$, with A^i the interior of the set A.

In the sequel of this section we assume that conditions S and C are satisfied.

PROPOSITION 1. *If* $f \in C(A)$ *then* $G_A f \in C(A)$ *and hence the embedded Markov process on A is stable.*

PROOF. According to a well-known theorem on weak convergence of probability measures [Billingsley (1968), p.12] it is sufficient to show that for any nonnegative lower semicontinuous (l.s.c.) function g it holds that $G_A I_A g$ is l.s.c. It follows from condition C that if g is l.s.c. then $PI_A g$ is l.s.c. The compact set A is closed and hence 1_{A^c} (the indicator function of A^c) is l.s.c. Consequently if g is l.s.c. then $PI_{A^c} g$ is l.s.c. Combining these arguments we find that $\sum_{n=0}^{N} P_A^n I_A g$ is l.s.c. for N=1,2,.... Since the limit of a nondecreasing sequence of l.s.c. functions is also l.s.c., we obtain that $G_A I_A g$ is l.s.c. To prove the second part of the assertion we note that the transition probabilities of the Markov process on A equal

$$_A P(x,B) = G_A \, 1_B(x),$$

with $x \in A$ and 1_B the indicator function of the set $B \subset A$, $B \in F$. Interpreting the following terms as the probabilities that the Markov process does and does not visit the set A before time n+1, shows

$$\sum_{k=1}^{n} P_A^k 1_A + P_A^n 1_{A^c} = 1_E.$$

In the first part of the proof of proposition 3 we shall show that $G_A 1_E(x) < \infty$ for all $x \in E$. Hence $\lim_{n\to\infty} P_A^n 1_{A^c} = 1_\emptyset$ and consequently $G_A 1_A = 1_E$. □

PROPOSITION 2. *The embedded Markov process on* A *has an invariant probability measure.*

PROOF. The assertion follows immediately from the well-known fact that a stable process on a compact state space has an invariant probability measure (see for example [Rosenblatt (1971), p.99]). To be complete we give an elementary proof of this result. Define for fixed $x \in A$,

$$\Pi_N(B) = \frac{1}{N} \sum_{n=1}^{N} {}_A P^n(x,B), \qquad N=1,2,\ldots .$$

It follows from a well-known theorem of Prohorov (cf.[Billingsley (1968), p. 37]) that (Π_N) , $N=1,2,\ldots$, has a weakly convergent subsequence. Hence for some subsequence N_k, $k=1,2,\ldots$, and some probability measure Π on A we have that

(2.1) $$\lim_{k\to\infty} \Pi_{N_k} g = \Pi g \qquad \text{for all } g \in C(A).$$

It is easily seen that also

(2.2) $$\lim_{k\to\infty} \Pi_{N_k}\, {}_A P g = \Pi g \qquad \text{for all } g \in C(A).$$

If $g \in C(A)$ then according to proposition 1 also ${}_A Pg \in C(A)$ and hence from (2.1)

$$\lim_{k\to\infty} \Pi_{N_k} \, {}_A Pg = \Pi_A Pg. \tag{2.3}$$

Combining (2.2) and (2.3) we find that

$$\Pi g = \Pi_A Pg \qquad\qquad \text{for all } g \in C(A).$$

Consequently, Π is an invariant probability measure (cf. [Billingsley (1968), theorem 1.3 on p. 9]). $\square$

PROPOSITION 3. *The Markov process has an invariant probability measure.*

PROOF. We first show that $G_A \, 1_E(x)$ is bounded on A. From the first part of the condition S we have that

$$1_E + PI_{A^c} \, \phi \le \phi \qquad\qquad \text{on } A^c.$$

Iterating this inequality N times we obtain

$$\sum_{n=0}^{N} (PI_{A^c})^n \, 1_E + (PI_{A^c})^{N+1} \, \phi \le \phi \qquad \text{on } A^c.$$

Hence

$$G_A \, 1_E \le \phi \qquad\qquad \text{on } A^c.$$

From the second part of condition S it follows that

$$G_A \, 1_E = P \, 1_A + P \, I_{A^c} \, G_A \, 1_E \le$$

$$\leq P\, 1_A + P\, I_{A^c}\, \phi,$$

is bounded on A.

Let Π_A be an invariant probability measure for the embedded Markov process on A. Then we define a measure Π on E by

$$\Pi\, 1_B = \Pi_A\, G_A\, 1_B.$$

Since $\Pi_A\, G_A\, 1_E$ is finite we have that Π is a finite measure.

Now we proceed along the same lines as in [Harris (1956).] to prove that Π is an invariant measure. Indeed, since $\Pi_A\, G_A\, 1_A = \Pi_A$ we have for $B \in \mathcal{F}$

$$\begin{aligned} \Pi\, P\, 1_B &= \Pi_A\, P\, 1_B + \Pi_A\, G_A\, I_{A^c}\, P\, 1_B = \\ &= \Pi_A\, [P\, 1_B + G_A\, I_{A^c}\, P\, 1_B] = \\ &= \Pi_A\, G_A\, 1_B = \Pi\, 1_B. \end{aligned}$$

Finally $(\Pi 1_E)^{-1}\, \Pi(.)$ is an invariant probability measure. □

3. QUEUEING PROCESSES

Following [Lindley (1952)] (see also [Feller (1966), p. 194]) we define recursively a sequence of random variables $W_0, W_1, \ldots$ by $W_0 = 0$ and

$$W_{n+1} = \max\, [W_n + U_{n+1}, 0], \qquad n=0,1,\ldots\, ,$$

where U_n denotes the difference of the $(n-1)^{th}$ service time and the n^{th} interarrival time. In [Lindley (1952)] it is assumed that the random variables U_n are i.i.d. Here we allow that U_{n+1} depends on W_n, for example the service time of the n^{th} customer depends on his waiting time.

ASSUMPTIONS. *The conditional distribution of* U_{n+1} *given* $W_n=w$ *does not depend on* n *(let* $F_w(.)$ *denote the regular version of* U_{n+1} *given* $W_n=w$*). For* $f \in C([0,\infty))$ *it holds that*

$$(3.1) \qquad \int_{-x}^{y-x} f(x+z)\ dF_x(z) + f(0)\ F_x(-x)$$

is continuous in x *for* y *sufficiently large and for* $y = \infty$*. And, moreover,*

$$(3.2) \qquad \limsup_{x\to\infty} \int_{-x}^{\infty} z\ \ dF_x(z) = -\ a < 0.$$

The stochastic kernel corresponding to the Markov process W_n, n=0,1,..., satisfies

$$P(x,[0,y]) = F_x(y-x).$$

Let y be such that for $x \geq y$

$$\int_{-\infty}^{\infty} (x+z)\ \ dF_x(z) \leq x \int_{-x}^{\infty} dF_x(z) + \int_{-x}^{\infty} z\ dF_x(z) \leq x - a/2$$

and, moreover, (3.1) be satisfied, then it is straightforward to verify that conditions S and C of section 2 hold with $A = [0,y]$, $\phi(x) = 2x/a$. It is well-known that the Doeblin condition implies the existence of an invariant probability measure. However, as pointed out in [Runnenburg (1960), p. 33], very few queueing processes satisfy the condition of Doeblin. If for some indecomposable Markov process the Doeblin condition holds then condition S is satisfied for a bounded function ϕ (cf. [Orey (1971]).

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